

## ON THE SINGULAR SETS OF A MODULE II

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Throughout this note,  $A$  and  $B$  will denote a (non-trivial) commutative Noetherian ring with a multiplicative identity element and  $M$  will denote a non-zero finitely generated  $A$ -module.

For every non-negative integer  $k$ , the set  $S_k^*(M) = \{p \in \text{Spec}(A) \mid \text{depth } M_p + \dim A/p \leq k\}$  is called the **singular set of  $M$**  with respect to  $k$ .

It is known that when the ring  $A$  is homomorphic image of a biequidimensional regular ring, then the singular sets of  $M$  are all closed in the Zariski topology on  $\text{Spec}(A)$  (see [3; ch. IV, 5]).

A development of this famous theorem has been recently shown in the sense that if  $A$  is a homomorphic image of a biequidimensional Gorenstein ring, the singular sets of  $M$  are still closed (See [2]).

The purpose of this article is to show that if  $B$  homomorphic image of a Cohen-Macaulay local ring then  $S_k^*(N)$  is closed, for every finitely generated  $B$ -module  $N$ .

First we prove some preliminary lemmas and propositions which help us to conclude the subsequent main theorem. From now on,  $A$  will denote a Cohen-Macaulay local ring with the unique maximal ideal  $m$ , and  $\hat{A}$  (respectively  $\hat{M}$ ) will denote the completion of  $A$  (respectively  $M$ ).

**1. Proposition.** Let  $\phi : A \rightarrow \hat{A}$  be the natural homomorphism. Then for every  $q \in \text{Spec}(\hat{A})$ ,  $S_k^*(\hat{M}) \cap q^c = S_k^*(M) \cap q^c$  (for any ideal  $J$  we write  $J^c$  for  $\phi^{-1}(J)$ ).

**Proof.** By [5; 23.3],  $\text{depth}_{A_q}(M_p \otimes_{A_p} \hat{A}_q) = \text{depth}_{\hat{A}_q}(\hat{M}_p)$ .

$\text{depth}_{\Lambda_p}(M_p) + \text{depth}(\hat{A}_q/pA_p\hat{A}_q)$ , since.

$$\begin{aligned} \bar{\varphi} : A_p &\rightarrow \hat{A}_q \\ \frac{a}{s} &\rightarrow \frac{\varphi(a)}{\varphi(s)} \end{aligned}$$

is a flat homomorphism. Also we have

$$\begin{aligned} M_p \otimes_{\Lambda_p} \hat{A}_q &\cong (M \otimes_{\Lambda} A_p) \otimes_{\Lambda_p} \hat{A}_q \cong M \otimes_{\Lambda} (A_p \otimes_{\Lambda_p} \hat{A}_q) \\ &\cong M \otimes_{\Lambda} \hat{A}_q \cong M \otimes_{\Lambda} (\hat{A} \otimes_{\Lambda} \hat{A}_q) \cong (M \otimes_{\Lambda} \hat{A}) \otimes_{\Lambda} \hat{A}_q \\ &\cong \hat{M} \otimes_{\Lambda} \hat{A}_q \cong \hat{M}_q. \end{aligned}$$

Thus we conclude that

$$\text{depth}_{\hat{\Lambda}_q}(\hat{M}_q) = \text{depth}_{\Lambda_p} M_p + \text{depth}(\hat{A}_q/pA_p\hat{A}_q).$$

On the other hand, since  $A$  is Cohen-Macaulay,  $\hat{A}$  is a Cohen-Macaulay local ring; whence, by corollary of [5;23.3],  $\hat{A}_q/pA_p\hat{A}_q$  is a Cohen-Macaulay ring. But

$$\hat{A}_q/pA_p\hat{A}_q = \hat{A}_q/p\hat{A}_q.$$

Hence

$$\text{depth}\left(\frac{\hat{A}_q}{pA_p\hat{A}_q}\right) = \dim\left(\frac{\hat{A}_q}{p\hat{A}_q}\right).$$

Moreover, by [5;15.1],

$$\text{ht } q = \text{ht } p + \dim\left(\frac{\hat{A}_q}{p\hat{A}_q}\right).$$

Hence

$$\text{depth}_{\hat{\Lambda}_q}(\hat{M}_q) = \text{depth}_{\Lambda_p}(M_p) + \text{ht}_q - \text{ht}_p.$$

From which we get, by [5;17.4],

$$\begin{aligned} \text{depth}_{\hat{\Lambda}_q}(\hat{M}_q) + \dim\left(\frac{\hat{A}}{q}\right) &= \text{depth}_{\Lambda_p}(M_p) + \dim \hat{A} - \text{ht } p \\ &= \text{depth}_{\Lambda_p}(M_p) + \dim A - \text{ht } p \\ &= \text{depth}_{\Lambda_p}(M_p) + \dim\left(\frac{A}{q}\right) \end{aligned}$$

The result now follows.

**2. Proposition.** With the same assumption as in Proposition 1. Let  $p, \hat{p} \in \text{Spec}(A)$  be prime ideals such that  $p \subseteq \hat{p}$  and  $p \in S_k^*(M)$ . Then  $\hat{p} \in S_k^*(M)$ .

**Proof.** Since  $\varphi: A \rightarrow \hat{A}$  is a faithfully flat homomorphism, there exists  $\hat{q} \in \text{Spec}(\hat{A})$  for which  $(\hat{q})^c = \hat{p}$  (by [5;7.3]). But  $\varphi$  has the going down property (see[5;9.5]). Hence there is a prime ideal

$q \in \text{Spec}(\hat{A})$  such that  $q^c = p$  and  $q \subseteq \hat{q}$ . By Proposition 1, this implies that  $q \in S_k^*(\hat{M})$ . But  $\hat{A}$  is a homomorphic image of a regular local ring (see[5;29.4(ii)]); thus by [3],  $S_k^*(\hat{M})$  is a closed subset of  $\text{Spec}(\hat{A})$  (note that, every Cohen-Macaulay local ring is biequidimensional ring). This implies that  $\hat{q} \in S_k^*(\hat{M})$ . Again from Proposition 1, this in turn implies that  $(\hat{q})^c = \hat{p} \in S_k^*(M)$  as required.

**3. Lemma.** (See[4;ch.1, §6, Ex. 1]) Let  $R \subseteq T$  be rings and  $p$  a minimal prime ideal in  $R$ . Then there exists in  $T$  a prime ideal contracting to  $p$ .

**Proof.** Let  $p$  be a minimal prime ideal of  $R$ . Set  $S = R/p$ , and

$$K = \{a \mid a \cap S = \phi \text{ \& } a \text{ is an ideal of } T\}.$$

Then  $K$  have a maximal element which is prime ideal of  $T$ . Let  $q$  be such prime ideal. Since  $(q \cap R) \cap S = \phi$ , we have  $(q \cap R) \subseteq p$  and consequently  $q \cap R = p$ .

We now turn to the main theorem of the note.

**4. Theorem.** For every positive integer  $k$ ,  $S_k^*(M)$  is a closed subset of  $\text{Spec}(A)$ .

**Proof:** Since  $S_k^*(\hat{M})$  is closed in  $\text{Spec}(\hat{A})$ , there exists an ideal  $J$  of  $\hat{A}$  such that  $V(J) = S_k^*(\hat{M})$ . It is enough to show that

$$V(J^c) = S_k^*(M)$$

Let  $p \in S_k^*(M)$ . Hence there is  $q \in S_k^*(\hat{A})$  such that  $q^c = p$ . Hence  $q \in S_k^*(\hat{M})$ . Thus  $J \subseteq q$ ; this implies that  $J^c \subseteq q^c = p$ ; i. e.,  $p \in V(J^c)$ .

Now let  $\hat{p} \in V(J^c)$ .  $\varphi$  induces the one-to-one homomorphism



$$\tilde{\varphi} : A/J^c \rightarrow \hat{A}/J$$

$$a + J^c \rightarrow \varphi(a) + J.$$

There is also a minimal prime ideal of  $J^c$  as  $p$  such that

$$J^c \subseteq p \subseteq \hat{p}.$$

Now by Lemma 3, there is  $q/J$  in  $\text{Spec}(\hat{A}/J)$  such that

$$\tilde{\varphi}^{-1}(q/J) = p/J^c$$

Hence  $p = q^c$  and  $J \subseteq q$ . Hence  $q \in V(J) = S_k^*(\hat{M})$ . It follows from Proposition 1 that  $\hat{p} \in S_k^*(M)$ . By Proposition 2, we conclude that  $\hat{p} \in S_k^*(M)$ .

Hence  $V(J^c) = S_k^*(M)$  and  $S_k^*(\hat{M})$  is closed as claimed.

**5. Corollary.** Let  $B$  be a homomorphic image of  $A$ . Then for every finitely generated  $B$ -module  $N$  the singular sets  $S_k^*(N)$  are closed.

**Proof.** Let  $f: A \rightarrow B$  be the relevant ring epimorphism. By [1;5], for every non-negative integer  $k$ ,

$$S_k^*(N) = \{p \in \text{Spec}(B) : f^{-1}(p) \in S_k^*(N|_A)\}$$

in which  $N|_A$  is the module  $N$  to be considered by restriction of scalars by means of  $f$ . Since  $S_k^*(N|_A)$  is a closed subset of  $\text{Spec}(A)$ , and  $f^*: \text{Spec}(B) \rightarrow \text{Spec}(A)$  is a continuous map, we conclude that  $f^{*-1}S_k^*(N|_A) = S_k^*(N)$  is a closed subset of  $\text{Spec}(B)$ .

## References

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